Calculus

B.Sc. 2nd Semester (FYUGP)

Unit 3 and Unit 4

Unit 3: Rolle's theorem, Lagrange's mean value theorem with geometrical interpretations and simple applications, Maclaurin and Taylor polynomials and their sigma notations. Taylor's formula with remainder, Introduction to Maclaurin and Taylor series.

Unit 4: Functions of two or more variables, Partial differentiation up to second order, Euler's theorem on homogeneous functions

Unit - 3

* Rolle's Theorem:

If a function $f(x)$ is derivable in an interval $[a,b]$ and also $f(a)=f(b)$, then there exist at least one value 'c' of x lying within $[a,b]$ such that $f'(c)=0$

(Or)

~~Proof:~~

If $f(x)$ is a function define on $[a,b]$ such that (i) $f(x)$ is continuous in $[a,b]$

(ii) $f(x)$ is differentiable in (a,b)

(iii) $f(a)=f(b)$

Then there exist one point $c \in (a,b)$ such that $f'(c)=0$

Proof: Given function $f(x)$ is continuous in $[a,b]$. Therefore, $f(x)$ attains its maximum (M) and minimum (m) value. Two cases arises; either $M=m$ or $M \neq m$

Case(i). $M=m$

$\Rightarrow$ $f(x)$ is a constant function

i.e, $f(x) = \text{constant}$

$\therefore$ $f'(x) = 0$

Case (ii). $m \neq M$

Since $f(a)=f(b)$,

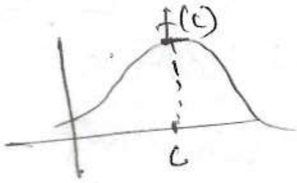
$\Rightarrow$ f has at least one value different from $f(a)$ and $f(b)$

Let $f(c) = M$, where $c \in (a,b)$

Also, f is differentiable in $[a,b]$

$\therefore f'(c)$ exist

i.e, $f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ will exist —— ①



Now, $f(c) = M$

$\Rightarrow f(c+h) \leq f(c)$

$\Rightarrow f(c+h) - f(c) \leq 0$ —— ②

When $h > 0$, from ① and ②, we have

$f'(c) \leq 0$ —— ③

When $h < 0$, from ① and ②, we have

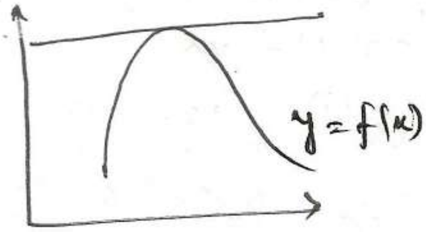
$f'(c) \geq 0$ —— ④

From ③ and ④, we get

$f'(c) = 0$

* Geometrical Interpretations of Rolle's Theorem:

Let the curve $y = f(x)$ which is continuous on $[a,b]$ and derivable on (a,b) be drawn.



The theorem simply states that between two points with equal ordinates on the graph of f, there exist atleast one point where the tangent is parallel to x-axis.

Algebraic: Between two zeros a and b of $f(x)$ (i.e, between two roots a and b of $f(x) = 0$) $\exists$ at least one zero of $f'(x)$.

Ex ① Verify Rolle's Theorems

(i) x^2 in $[-1,1]$ (ii) $x(x+3)e^{-\frac{1}{2}x}$ in $[-3,0]$

Proof (i) Let $f(x)=x^2$

Since the given function is polynomial function

$\therefore f(x)=x^2$ is continuous on $[-1,1]$

Also $f(x)=x^2$ is differentiable in $(-1,1)$

Now, $f(-1)=(-1)^2=1$

$f(1)=1^2=1$

$\therefore f(-1)=f(1)$

We have, $f'(x)=2x$

If $f'(x)=0 \Rightarrow 2x=0 \Rightarrow x=0$

Which value lies within $[-1,1]$

Hence the Rolle's Theorem is verified.

(ii) Let $f(x)=x(x+3)e^{-\frac{x}{2}}$

we have, $f(-3)=0=f(0)$

$f(x)$ is derivable in the interval $[-3,0]$.

We have, $f'(x)=(2x+3)e^{-\frac{x}{2}}+x(x+3)e^{-\frac{x}{2}}\left(-\frac{1}{2}\right)$

$$=\frac{-x^2+x+6}{2}e^{-\frac{x}{2}}$$

Putting $f'(x)=0$ we get $-x^2+x+6=0 \Rightarrow x=-2,3$

Of these two values of x, for which $f'(x)$ is zero, -2 belongs of the interval $[-3,0]$ under consideration.

Ex ②: Verify Rolle's theorem in the interval $[a,b]$ for the functions

(i) $\log \dfrac{x^2+ab}{(a+b)x}$

(ii) $(x-a)^m (x-b)^n$; m,n being +ve integer.

Solution: (i) Let $f(x) = \log \dfrac{x^2+ab}{(a+b)x}$

Since, $f(x)$ is continuous on $[a,b]$ and differentiable in (a,b).

Also, $f(a) = \log \dfrac{a^2+ab}{(a+b)a} = \log \dfrac{a(a+b)}{a(a+b)} = \log 1 = 0$

$f(b) = \log \dfrac{b^2+ab}{(a+b)b} = 0$

$\therefore f(a) = f(b)$

Now, $f'(x) = \dfrac{1}{\dfrac{x^2+ab}{(a+b)x}} \cdot \dfrac{(a+b)x(2x) - (x^2+ab)(a+b)}{(a+b)^2x^2}$

$= \dfrac{(a+b)x}{(x^2+ab)} \times \dfrac{(a+b)(2x^2-x^2-ab)}{(a+b)^2x^2}$

$= \dfrac{x^2-ab}{(x^2+ab)x}$

If $f'(x) = 0 \Rightarrow x^2 - ab = 0 \Rightarrow x = \pm\sqrt{ab}$

Since, $x = \sqrt{ab} \in (a,b)$

Hence, the Rolle's theorem is verified.

(ii) Try yourself.

[Hint: $f'(x) = 0$ -

$\Rightarrow x = \dfrac{an-bm}{n-m} \in (a,b)$]

Since, m,n is +ve integer.

* Lagrange's Mean Value Theorem:

If a function f defined on $[a, b]$ is

(i) continuous on $[a, b]$ and

(ii) derivable on (a, b)

Then $\exists$ at least one real number c between a and b such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \quad c \in (a, b)$$

Proof: Let us consider a function

$$\phi(x) = f(x) + Ax, \quad x \in [a, b]$$

where A is constant to be determined such that

$$\phi(a) = \phi(b)$$

$$\Rightarrow A = -\frac{f(b) - f(a)}{b - a}$$

Now, the function $\phi(x)$, being the sum of two continuous and derivable functions, is itself

(i) continuous on $[a, b]$ (ii) derivable on (a, b) and

(iii) $\phi(a) = \phi(b)$

Therefore, by Rolle's Theorem $\exists$ a real number $c \in (a, b)$ such that $\phi'(c) = 0$

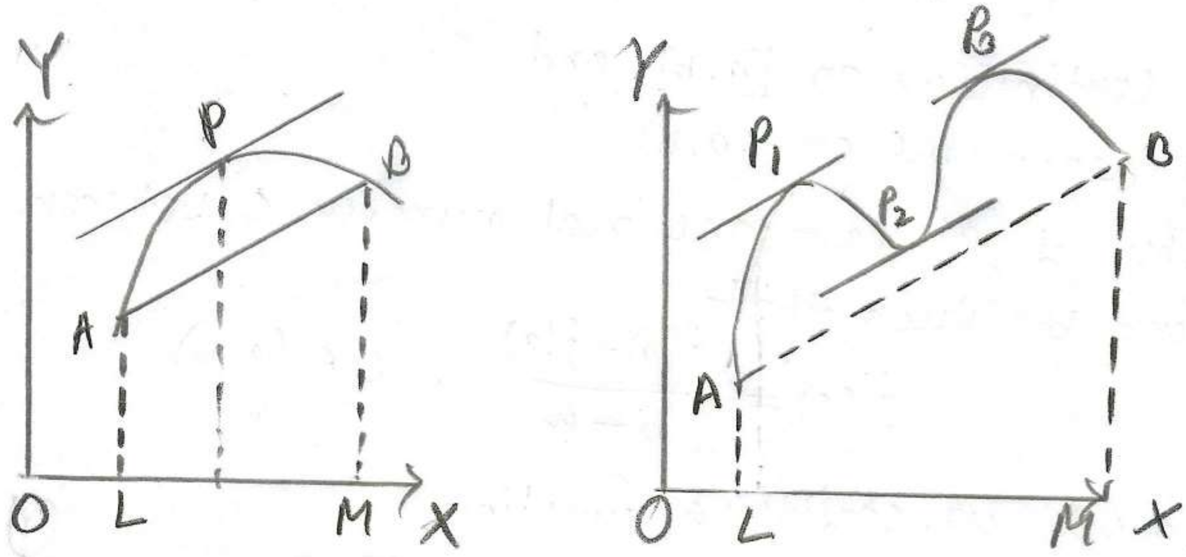
But $\phi'(x) = f'(x) + A$

$$\therefore \quad f'(c) + A = 0$$

$$\Rightarrow f'(c) = -A$$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Geometrical statement of the theorem :



If a curve has a tangent at each of its points, then there exist at least one point P on the curve such that the tangent P is parallel to the chord AB joining its extremities.

If $f(x)$ is derivable in $[a,b]$, then the curve $y=f(x)$ has a tangent at each point of the curve lying between the extremities $A[a,f(a)]$ and $B[b,f(b)]$

To slope of the chord $AB = \dfrac{f(b)-f(a)}{b-a}$

Let P be the point $(c, f(c))$ on the curve ; c being such that $\dfrac{f(b)-f(a)}{b-a} = f'(c)$.

The slope of the tangent at $P = f'(c)$

Thus $\exists$ a point P on the curve the tangent at which is parallel to the chord AB.

Ex:① If $f(x) = \frac{(x-1)(x-2)(x-3)}{1}$; $a = 0, b = 4$ then find the value of c.

Solution: Given, $f(x) = (x-1)(x-2)(x-3)$

$\therefore f(a) = f(0) = -6, \quad f(b) = f(4) = 3.2.1 = 6$

$$\therefore \frac{f(b) - f(a)}{b-a} = \frac{f(4) - f(0)}{4-0} = \frac{12}{4} = 3$$

Also, $f'(x) = (x-1)(x-2) + (x-2)(x-3) + (x-3)(x-1)$

$= x^2 - 3x + 2 + x^2 - 5x + 6 + x^2 - 4x + 3$

$= 3x^2 - 12x + 11$

By Lagrange's Mean Value Theorem

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$\Rightarrow 3c^2 - 12c + 11 = 3$

$\Rightarrow 3c^2 - 12c + 8 = 0$

$\Rightarrow c = \frac{6 + 2\sqrt{3}}{3}, \frac{6 - 2\sqrt{3}}{3} \in (0, 4)$

Ex(2): Verify the mean value theorem for

(i) $\log x$ in $[1, e]$ (ii) x^3 in $[a, b]$

(iii) $lx^2 + mx + n$ in $[a, b]$

Solution: (i) Let $f(x) = \log x$, $[1, e]$

~~Now, $f(1) = \log 1 = 0$~~

~~ff~~

Since Given function is logarithm functions.

$\therefore$ $f(x)$ is continuous on $[1, e]$

and $f(x)$ is differentiable in $(1, e)$

And, $f(a) = f(1) = \log 1 = 0$

$f(b) = f(e) = \log e = 1$

$\therefore \frac{f(b)-f(a)}{b-a} = f'(c) \Rightarrow \frac{1}{e-1} = f'(c)$ ——— (I)

Now, $f'(x) = \frac{1}{x} \Rightarrow f'(c) = \frac{1}{c}$ ——— (II)

From (I) and (II), we have

$\frac{1}{c} = \frac{1}{e-1} \Rightarrow c = e-1 \in [1, e)$

Since, $2 < e < 3$

Hence, $f(x) = \log x$ in $[1, e]$ is verified.

(ii) Try yourself

[Hint: $c^2 = \frac{a^2+ab+b^2}{3}$

Let $c_1 = \left(\frac{a^2+ab+b^2}{3}\right)^{1/2}$, $c_2 = -\left(\frac{a^2+ab+b^2}{3}\right)^{1/2}$

We need to prove that; at least one of the following two inequalities holds;

(I) $a < c_1 < b$; (II) $a < c_2 < b$

Consider 2 cases:

1. $0 \leq a < b$: In this case (I) is true because $c_1^2 - a^2 = \frac{1}{3}(b-a)(b+2a) > 0$ and

$b^2 - c_1^2 = \frac{1}{3}(b-a)(2b+a) > 0$

2. $a < b \leq 0$; similarly, in this case (II) is true because $a^2 - c_2^2 > 0$ and $c_2^2 - b^2 > 0$

Ex:3: Find c of the mean value theorem if
$$f(x) = x(x-1)(x-2) \;;\; a=0,\ b=\tfrac{1}{2}$$

Ans: $c = \dfrac{1-\sqrt{21}}{6}$

Ex 4: Find c so that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$ in following cases:—

(i) $f(x) = x^2 - 3x - 1 \;;\; a = \frac{-11}{7},\ b = \frac{13}{7}$

(ii) $f(x) = \sqrt{(x^2-4)} \;;\; a=2,\ b=3$

(iii) $f(x) = e^x \;;\; a=0,\ b=1$

Ans: (i) $c = \frac{1}{7} \in (a,b)$

(ii) $c = \sqrt{5} \in (a,b)$

(iii) $c = \log(e-1) \in (a,b)$

Ex. 5: Explain the failure of theorem in the interval $[-1,1]$ when $f(x) = \frac{1}{x},\ x \neq 0 \;;\; f(0)=0$

Solution: Let $f(x) = \frac{1}{x}$, $[-1,1]$

$$f'(x) = -\frac{1}{x^2}$$

Now,
$$f'(c) = \frac{f(b)-f(a)}{b-a}$$
$$\Rightarrow -\frac{1}{c^2} = \frac{1-(-1)}{1-(-1)} = 1$$
$$\Rightarrow c^2 = -1$$
$$\Rightarrow c = \pm\sqrt{-1} \notin [-1,1]$$

Therefore, Mean value theorem cannot applied.

* Some Important deductions from the Mean Value Theorem :

Let x_1, x_2 be any two points belonging to the interval $[a, b]$ such that $x_1 < x_2$.

Applying the mean value theorem to the interval $[x_1, x_2]$. We see that $\exists$ a number q between x_1 and x_2 such that

$$f'(q) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad \text{—— (1)}$$

(A) Let $f'(x) = 0$ throught the interval $[a, b]$.

From (1), we get $f(x_2) = f(x_1)$

Where, x_1, x_2 are any two values of x, Thus we see that every two values of the function are equal. Hence $f(x)$ is a constant. We thus proved

" If the derivative of a function vanishes for all of x in an interval, then the function must be constant."

(B) Let $f'(x) > 0$ for every value of x in $[a, b]$

From (1), we get

$$f(x_2) - f(x_1) > 0 \Rightarrow f(x_2) > f(x_1)$$

Hence $f(x)$ is an increasing function of x.

We thus proved —

" A function whose derivative is positive for every values of x in an interval is a monotonically increasing function of x in that interval".

(c) Let $f'(x) < 0$ for every value of x in $[a, b]$

From ①, we have $f(x_2) - f(x_1) < 0$

$$\Rightarrow f(x_2) < f(x_1)$$

Hence, $f(x)$ is a decreasing function of x.

We have thus proved :—

"A function whose derivative is negative for every value of x in an interval is a monotonically decreasing function of x in that interval."

Examples

① Show that $x^3 - 3x^2 + 3x + 2$ is monotonically increasing in every interval.

Solution: Let $f(x) = x^3 - 3x^2 + 3x + 2$

$\therefore f'(x) = 3x^2 - 6x + 3 = 3(x-1)^2$

Thus, $f'(x) > 0$ for every value of x except 1 where it vanishes.

Hence $f(x)$ is monotonically increasing.

② Separate the interval in which the polynomial

$$2x^3 - 15x^2 + 36x + 1$$

is decreasing or increasing.

Also draw a graph of the function.

Solution: Let $f(x) = 2x^3 - 15x^2 + 36x + 1$

$\therefore f'(x) = 6x^2 - 30x + 36 = 6(x-2)(x-3)$

To find interval, we consider

$f'(x)=0 \Rightarrow 6x^2-30x+36=0$

$\Rightarrow x^2-5x+6=0$

$\Rightarrow x=2,3$

$\therefore$ Intervals are $(-\infty,2)$, $[2,3]$, ~~and~~ $(3,\infty)$

We have, $f'(x)=0$ for $x=2$ and 3.

$\Rightarrow f(x)=$ constant.

Again, Now,

For $x<2$, $f'(x)>0$

For $2<x<3$, $f'(x)<0$

For $x>3$, $f'(x)>0$

Thus, $f'(x)$ is positive in $(-\infty,2)$ and $(3,\infty)$ and negative in the interval $(2,3)$

Hence, $f(x)$ is monotonically increasing in the intervals $(-\infty,2)$, $[3,\infty)$ and monotonically decreasing in $[2,3]$.

(5) Show that $\frac{x}{1+x}<\log(1+x)<x$ for $x>0$ (VVI)

Solution: Let $f(x)=\log(1+x)-\frac{x}{1+x}$ $\therefore f'(x)=\frac{x}{(1+x)^2}$

Thus, $f'(x)>0$ for $x>0$ and $f'(x)=0$ for $x=0$

Hence, $f(x)$ is monotonically increasing in the interval $[0,\infty)$. Also, $f(0)=0$, $\therefore f(x)>f(0)=0$ for $x>0$.

Hence, $f(x)$ is positive for every positive value of x, so that $\log(1+x)>\frac{x}{1+x}$, for $x>0$. ——(1)

Again, Let $F(x)=x-\log(1+x)$

$\therefore F'(x)=1-\frac{1}{1+x}=\frac{x}{1+x}$

Thus $F'(x)>0$ for $x>0$ and $F'(x)=0$ for $x=0$

Therefore, $F(x)$ is monotonically increasing in the interval $[0, \infty)$. Also $F(0) = 0$

$$F(x) > F(0) = 0, \text{ for } x > 0$$

Hence $F(x)$ is positive for positive values of x so that $x > \log(1+x)$, for $x > 0$ —— (ii)

From (i) and (ii), we have

$$\frac{x}{1+x} < \log(1+x) < x, \text{ for } x > 0$$

<u>Ex:(4)</u> Show that $x < -\log(1-x) < x(1-x)^{-1}$, for $0 < x < 1$

<u>Ex(5):</u> Determine the intervals in which the function $(x^4 + 6x^3 + 17x^2 + 32x + 32)e^{-x}$ is increasing or decreasing.

<u>Ex(6):</u> Let $f(x) = ax^2 + bx + c$. Let p and q are two real members such that $f(p) = f(q)$.

prove that $f'\left(\frac{p+q}{2}\right) = 0$

<u>Solution:</u> Given function $f(x) = ax^2 + bx + c$.

Since the given function is a polynomial function. So, it is continuous and differentiable for every value of x, in particular in $[p, q]$

Also, given that $f(p) = f(q)$

Thus, the given function f satisfies all the conditions of Rolle's Theorem, hence there is at least one value in say c

$$f'(c) = 0 \quad \text{———} \quad (1)$$

$$\Rightarrow 2ac + b = 0$$

$$\Rightarrow c = -\frac{b}{2a}, \quad c \in (p, q) \quad \text{———} \quad (11)$$

But, $f(p) = f(q)$

$$\Rightarrow ap^2 + bp + c = aq^2 + bq + c$$

$$\Rightarrow a(p^2 - q^2) + b(p - q) = 0$$

$$\Rightarrow a(p+q)(p-q) + b(p-q) = 0$$

$$\Rightarrow a(p+q) + b = 0$$

$$\Rightarrow p + q = -\frac{b}{a}$$

$$\Rightarrow \frac{p+q}{2} = -\frac{b}{2a} \quad \text{———} \quad (111)$$

From (11) and (111), we have

$$c = \frac{p+q}{2}$$

using the values of c in (1), we have

$$f'\left(\frac{p+q}{2}\right) = 0.$$

* Maclaurin and Taylor Polynomial :

(1) Taylor polynomial: If f can be differentiated n times at x_0, then we define the nth Taylor polynomial for f about $x = x_0$ to be

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \dots + \frac{f^n(x_0)}{n!}(x - x_0)^n.$$

(2) Maclaurin polynomial: If f can be differentiated n times at 0, then we define the nth Maclaurin polynomial for f to be

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n$$

Example:

(1) Find the Maclaurin polynomial P_0, P_1, P_2, P_3 and P_n for e^x.

Solution: Given function $f(x) = e^x$

We know that Maclaurin polynomial

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n$$

Since, $f(x) = e^x$, $f(0) = e^0 = 1$

$f'(x) = e^x$, $f'(0) = e^0 = 1$

$f''(x) = e^x$, $f''(0) = e^0 = 1$

$\vdots$

$f^{(n)}(x) = e^x$ $f^{(n)}(0) = e^0 = 1$

$\therefore$ Maclaurin polynomial

$P_0(x) = f(0) = 1$

$P_1(x) = f(0) + f'(0)x = 1 + x$

$P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = 1 + x + \frac{x^2}{2!}$

$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$

$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n$

$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$

Example: (2) Find the nth Maclaurin polynomial for

(a) $\sin x$ (b) $\cos x$

Solution: (a) $\sin x$

Given, $f(x) = \sin x$ $\Rightarrow f(0) = 0$

$\therefore$ $f'(x) = \cos x$, $f'(0) = 1$

$$f''(x) = -\sin x \quad , \quad f''(0) = 0$$

$$f'''(x) = -\cos x \quad , \quad f'''(0) = -1$$

$$f^{iv}(x) = \sin x \quad , \quad f^{iv}(0) = 0$$

The pattern 0, 1, 0, −1 will repeat as we evaluate the successive derivative at 0. Therefore, Maclaurin polynomial is

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(iv)}(0)}{4!}x^4 + \dots + \frac{f^{(n)}(0)}{n!}x^n$$

$$= 0 + x + \frac{0}{2!}x^2 + \frac{(-1)}{3!}x^3 + \frac{0}{4!}x^4 + \dots + \frac{(-1)^n x^n}{n!}$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + (-1)^n \frac{x^n}{n!} \quad \#$$

(OR)

$P_0(x) = 0$

$P_1(x) = 0 + x$

$P_2(x) = 0 + x + 0$

$P_3(x) = 0 + x + 0 - \frac{x^3}{3!}$

$P_4(x) = 0 + x + 0 - \frac{x^3}{3!} + 0$

$P_5(x) = 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!}$

$P_6(x) = 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0$

$P_7(x) = 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0 - \frac{x^7}{7!}$

* Sigma notation for Taylor and Maclaurin Polynomial:

To express Taylor and Maclaurin polynomial in Sigma notation, we use the notation $f^{(k)}(x_0)$ to denote the k^{th} derivative of f at $x = x_0$ and we make the convection that $f^{(0)}(x_0)$ denotes $f(x_0)$.

Thus Taylor polynomial becomes.

$$\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x-x_0)}{n!}(x-x_0)^n.$$

Similarly, we can write the Maclaurin polynomial

$$\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n.$$

Example: Find the nth Taylor polynomial for $\frac{1}{x}$ about $x = 1$ and express it in sigma notations.

Solution: Given function $f(x) = \frac{1}{x}$

At about $x = 1$, we have

$f(x) = \frac{1}{x}$, $\quad f(1) = 1$

$f'(x) = -\frac{1}{x^2}$, $\quad f'(1) = -1$

$f''(x) = \frac{2}{x^3}$, $\quad f''(1) = 2 = 2!$

$f'''(x) = -\frac{6}{x^4}$, $\quad f'''(1) = -6 = -3!$

$$f^{(4)}(x) = \frac{24}{x^5}, \quad f^{(4)}(1) = 24 = 4!$$

Similarly,

$$f^{(k)}(x) = \frac{k!}{x^{k+1}}, \quad f^{(k)}(1) = (-1)^k k!$$

$\therefore$ Taylor polynomial about $x = 1$ is

$$\sum_{k=0}^{n} \frac{(-1)^k k!}{k!}(x-1)^k = 1 + (-1)(x-1) + \frac{2!}{2!}(x-1)^2 + \frac{(-3!)}{3!}(x-1)^3 + \frac{4!}{4!}(x-1)^4 + \dots + \frac{(-1)^n n!}{n!}(x-1)^n$$

$$\Rightarrow \sum_{k=0}^{n} (-1)^k (x-1)^k = 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 + \dots + (-1)^n (x-1)^n.$$

Example: use sigma notation to write the Taylor series about $x = -1$ for the function e^x.

Solution: Given $f(x) = e^x$

At About $x = -1$, we have

$$f(x) = e^x, \quad f(-1) = e^{-1} = \frac{1}{e}$$

$$f'(x) = e^x, \quad f'(-1) = \frac{1}{e}$$

$$f''(x) = e^x, \quad f''(-1) = \frac{1}{e}$$

Similarly

$$f^{(n)}(x) = e^x, \quad f^{(n)}(-1) = \frac{1}{e}$$

$\therefore$ Taylor series for the function e^x about $x=-1$ in Sigma notation is given by

$$\sum_{k=0}^{n} \frac{\frac{1}{e}}{k!}(x-(-1))^k = \frac{1}{e} + \frac{1}{e}(x-(-1)) + \frac{1}{e2!}(x-(-1))^2 + \frac{1}{e3!}(x-(-1))^3 + \dots + \frac{1}{en!}(x-(-1))^n$$

$$\Rightarrow \sum_{k=0}^{n} \frac{1}{ek!}(x+1)^k = \frac{1}{e} + \frac{1}{e}(x+1) + \frac{1}{e2!}(x+1)^2 + \frac{1}{e3!}(x+1)^3 + \dots + \frac{1}{en!}(x+1)^n.$$

$\#$

* Taylor's formula with remainder:

It will be convenient to have a notations for the error in the approximation $f(x) \approx P_n(x)$. Accordingly, we will let $R_n(x)$ denote the difference between $f(x)$ and its n^{th} Taylor polynomial; that is

$$R_n(x) = f(x) - P_n(x)$$
$$= f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

This can also be written as

$$f(x) = P_n(x) + R_n(x)$$
$$= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + R_n(x)$$

The function $R_n(x)$ is called the n^{th} remainder for the Taylor series of f and above formula is called Taylor formula with remainder.

* Maclaurin and Taylor Series :

If f can be differentiated nth times at x_0, then Taylor series is given by

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

And, Maclaurin series is given by

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(k)}(0)}{k!}x^k$$

Example: Expand Maclaurin series of

(i) $\sin x$ (ii) $\cos x$ (iii) e^x (iv) $\log(1+x)$

Sol: (i) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + \dots$

(ii) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + (-1)^k \frac{x^{2k}}{(2k)!} + \dots$

(iii) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^k}{k!} + \dots$

(iv) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{k-1} \frac{x^k}{k} + \dots$

unit-4

* Functions of two or more Variable :

① A function f of two variables x and y is a rule that assign a unique real numbers $f(x,y)$ to each point (x,y) in some set D in the xy-plane. e.g,

$$f(x,y) = x^2 + y^2 - 1$$

$$f(x,y) = \sqrt{1-x^2-y^2}$$

② A function f of three variables x, y and z is a rule that assign a unique real numbers $f(x,y,z)$ to each point (x,y,z) in some set D in the xyz-plane. e.g,

$$f(x,y,z) = x+y+z$$

$$f(x,y,z) = 1-xyz$$

* Partial derivatives of function of two variables :

Let $z = f(x,y)$ Then

$\lim_{h \to 0} \frac{f(a+h,b)-f(a,b)}{h}$ if it exists, is said to be partial derivatives of $f(x,y)$ w.r.t 'x' at (a,b) and is denoted by

$$\left(\frac{\partial z}{\partial x}\right)_{(a,b)} \text{ or } f_x(a,b)$$, $\partial \to$ del

Again, let $z = f(x,y)$ then

$\lim_{k \to 0} \frac{f(a,b+k)-f(a,b)}{k}$ if it exists, is said partial derivatives of $f(x,y)$ w.r.t y at (a,b) and is denoted by $\left(\frac{\partial z}{\partial y}\right)_{(a,b)}$ or $f_y(a,b)$.

Note: If $f(x,y)$ possesses a partial derivatives w.r.t x and y at every point of its domain of definition, then it is denoted by

$$\frac{\partial f}{\partial x} \text{ or } f_x \text{ or } \frac{\partial z}{\partial x}$$

And, $\frac{\partial f}{\partial y}$, or f_y or $\frac{\partial z}{\partial y}$.

* partial derivatives of higher order:

We can form partial derivatives of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ w.r.t 'x'. We have

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right), \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

Which are called the second order partial derivatives of z and are denoted by

$$\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y} \text{ or } f_{xx}, f_{xy}$$

Ex① Find $f_x(1,3)$ and $f_y(1,3)$ for the function $f(x,y) = 2x^3y^2 + 2y + 4x$.

Solution: Given $f(x,y) = 2x^3y^2 + 2y + 4x$

[**Remember:** To find partial derivative of $f(x,y)$ w.r.t 'x' then we take 'y' as constant, and if w.r.t 'y' then we take 'x' as constant]

$$\therefore f_x(x, y) = \frac{\partial}{\partial x}\left(2x^3y^2 + 2y + 4x\right)$$
$$= 6x^2y^2 + 4$$
$$f_y(x, y) = \frac{\partial}{\partial y}\left(2x^3y^2 + 2y + 4x\right)$$
$$= 4x^3y + 2$$

At $(1,3)$

$$\therefore f_x(1,3) = 6.1^2.3^2 + 4 = 58$$
$$f_y(1,3) = 4.1^3.3 + 2 = 14$$

Ex 2: Find $f_x(x,y)$ and $f_y(x,y)$ for $f(x,y) = 2x^3y^2 + 2y + 4x$ and use those partial derivatives to compute $f_x(1,3)$ and $f_y(1,3)$. [Ans: Ex(1)]

Ex 3: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z = x^4 \sin(xy^3)$

Solution: Given $z = x^4 \sin(xy^3)$

$$\therefore \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left[x^4 \sin(xy^3)\right]$$
$$= x^4 \frac{\partial}{\partial x}\left[\sin(xy^3)\right] + \sin(xy^3)\frac{\partial}{\partial x}(x^4)$$
$$= x^4 \cos(xy^3)\frac{\partial}{\partial x}(xy^3) + \sin(xy^3) . 4x^3$$
$$= x^4y^3 \cos(xy^3) + 4x^3 \sin(xy^3)$$

Similarly $\frac{\partial z}{\partial y} = 3x^5y^2 \cos(xy^3)$.

Ex 4: If $f(x,y,z) = x^3y^2z^4 + 2xy + z$ then find $f_x(x,y,z)$, $f_y(x,y,z)$, $f_z(x,y,z)$

Solution: To differentiate partially w.r.t 'x' we take y and z as constant.

$\therefore f_x(x,y,z) = \frac{\partial}{\partial x}(x^3y^2z^4 + 2xy + z)$

$= 3x^2y^2z^4 + 2y$

$f_y(x,y,z) = 2x^3yz^4 + 2x$

$f_z(x,y,z) = 4x^3y^2z^3 + 1$

<u>Ex 5</u>: Let $f(x,y) = y^2e^x + y$. Find f_{xyy}

<u>Solution</u>: We know, $f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$

$\therefore f_{xyy} = \frac{\partial}{\partial y^2}\left(\frac{\partial f}{\partial x}\right)$

Now, $\frac{\partial f}{\partial x} = y^2e^x$

$\therefore \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 2ye^x$

$\frac{\partial}{\partial y^2}\left(\frac{\partial f}{\partial x}\right) = 2e^x$

Hence, $f_{xyy} = 2e^x$,

<u>Ex: 6</u> Find the first order partial derivatives of (i) $\tan^{-1}(x+y)$ (ii) $e^{ax}\sin by$ (iii) $\log(x^2+y^2)$

<u>Solution</u>: (i) $\tan^{-1}(x+y)$

$\frac{\partial}{\partial x}\left(\tan^{-1}(x+y)\right) = \frac{1}{1+(x+y)^2}\frac{\partial}{\partial x}(x+y) = \frac{1}{1+(x+y)^2}$

$\frac{\partial}{\partial y}\left(\tan^{-1}(x+y)\right) = \frac{1}{1+(x+y)^2}\frac{\partial}{\partial y}(x+y) = \frac{1}{1+(x+y)^2}$

(ii) $e^{ax}\sin by$

$\frac{\partial}{\partial x}\left(e^{ax}\sin by\right) = \sin by\frac{\partial}{\partial x}(e^{ax}) = e^{ax}\sin by$

$\frac{\partial}{\partial y}\left(e^{ax}\sin by\right) = e^{ax}\frac{\partial}{\partial y}(\sin by) = be^{ax}\cos by$

(iii) $\log(x^2+y^2)$

$$\frac{\partial}{\partial x}\left[\log(x^2+y^2)\right] = \frac{1}{x^2+y^2}\frac{\partial}{\partial x}(x^2+y^2) = \frac{2x}{x^2+y^2}$$

$$\frac{\partial}{\partial y}\left[\log(x^2+y^2)\right] = \frac{1}{x^2+y^2}\frac{\partial}{\partial y}(x^2+y^2) = \frac{2y}{x^2+y^2}$$

Ex 7: Find the second order derivatives of

(i) e^{x-y} (ii) e^{xy} (iii) $\tan(\tan^{-1}x + \tan^{-1}y)$

Solution: (i) $e^{x-y} = f$ (say)

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(e^{x-y}) = e^{x-y}\frac{\partial}{\partial x}(x-y) = e^{x-y}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(e^{x-y}) = e^{x-y}$$

Again, $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(e^{x-y}) = e^{x-y}\frac{\partial}{\partial y}(x-y) = -e^{x-y}$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(-e^{x-y}) = e^{x-y}$$

[(ii) and (iii) Try yourself]

Ex: 8 Verify that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

When u is (i) $\sin^{-1}\frac{x}{y}$ (ii) $\frac{xy}{\sqrt{1+x^2+y^2}}$

(iii) $\log(y\sin x + x\sin y)$.

Solution: (i) Given $u = \sin^{-1}\frac{x}{y}$

We want to show $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

Now, $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\sin^{-1}\frac{x}{y}$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \frac{\partial}{\partial x}\left(\frac{x}{y}\right) = \frac{1}{\sqrt{\left(\frac{y^2-x^2}{y^2}\right)}} \frac{1}{y} = \frac{1}{\sqrt{y^2-x^2}}$$

And, $\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\sin^{-1}\frac{x}{y}\right) = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \frac{\partial}{\partial y}\left(\frac{x}{y}\right)$

$$= \frac{1}{\sqrt{\frac{y^2-x^2}{y^2}}}\left(-\frac{x}{y^2}\right) = \frac{-x}{y\sqrt{y^2-x^2}}$$

Again, $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{-x}{y\sqrt{y^2-x^2}}\right)$

$$\Rightarrow \frac{\partial^2 u}{\partial x \partial y} = \frac{y\sqrt{y^2-x^2}\,\frac{\partial}{\partial x}(-x) - (-x)\frac{\partial}{\partial x} y\sqrt{y^2-x^2}}{y^2(y^2-x^2)}$$

$$= \frac{y\sqrt{y^2-x^2}\,(-1) + yx\frac{1}{2\sqrt{y^2-x^2}}(-2x)}{y^2(y^2-x^2)}$$

$$= \frac{-y(y^2-x^2) + xy(-x)}{y^2(y^2-x^2)^{3/2}}$$

$$= \frac{-y}{(y^2-x^2)^{3/2}} \quad \text{—— (i)}$$

And, $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{1}{\sqrt{y^2-x^2}}\right)$

$$= \frac{-1\,\frac{\partial}{\partial y}\left(\sqrt{y^2-x^2}\right)}{(y^2-x^2)}$$

$$= \frac{-\frac{1}{2\sqrt{y^2-x^2}}(2y)}{(y^2-x^2)}$$

$$= \frac{-y}{(y^2-x^2)^{3/2}} \quad \text{—— (ii)}$$

From (i) and (ii), we get $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

* Homogeneous Functions:

① A function $F(x,y)$ is said to be homogeneous function of degree n if

$$F(\lambda x, \lambda y) = \lambda^n F(x,y)$$

for any nonzero constant λ.

② A function $f(x,y)$ is a homogeneous function of order or degree n, if it is expressible as $x^n f\left(\frac{y}{x}\right)$ or $y^n f\left(\frac{x}{y}\right)$.

Example: ① $F_1(x,y) = y^2 + 2xy$ ⓘⓘ $F_2(x,y) = 2x - 3y$

(iii) $F_3(x,y) = \cos\left(\frac{y}{x}\right)$ (iv) $F_4(x,y) = \frac{\sqrt{y}+\sqrt{x}}{y+x}$

Explanation: ① $F_1(\lambda x, \lambda y) = (\lambda y)^2 + 2(\lambda x)(\lambda y)$

$$= \lambda^2 (y^2 + 2xy)$$

$$= \lambda^2 F_1(x,y)$$

So, $F_1(x,y)$ is a homogeneous function of degree 2.

(iii) $F_3(x,y) = \cos\left(\frac{y}{x}\right)$

$$\therefore F_3(\lambda x, \lambda y) = \cos\left(\frac{\lambda y}{\lambda x}\right) = \cos\left(\frac{y}{x}\right)$$

$$= \lambda^0 \cos\left(\frac{y}{x}\right)$$

So, $F_3(x,y)$ is a homogeneous function of degree 0.

(iv) $F_4(\lambda x, \lambda y) = \frac{\sqrt{\lambda y} + \sqrt{\lambda x}}{\lambda y + \lambda x} = \frac{\sqrt{\lambda}(\sqrt{y}+\sqrt{x})}{\lambda (y+x)} = \lambda^{-\frac{1}{2}} \left(F_4(x,y)\right)$

So, $F_4(x,y)$ is a homogeneous function of degree $-\frac{1}{2}$.

* Euler's Theorem on Homogeneous Function:

If f be a homogeneous function of x, y of degree n, then

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf.$$

<u>**Proof:**</u> Let f be a homogeneous function of x, y of degree n.

i.e. $f = x^n f\left(\frac{y}{x}\right)$ ——— (1)

$$\therefore \frac{\partial f}{\partial x} = nx^{n-1} f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)$$

$$= nx^{n-1} f\left(\frac{y}{x}\right) - yx^{n-2} f'\left(\frac{y}{x}\right)$$

and $\frac{\partial f}{\partial y} = x^n f'\left(\frac{y}{x}\right)\cdot\frac{1}{x}$

$$= x^{n-1} f'\left(\frac{y}{x}\right)$$

Now, $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}$

$$= nx^n f\left(\frac{y}{x}\right) - yx^{n-1} f'\left(\frac{y}{x}\right) + yx^{n-1} f'\left(\frac{y}{x}\right)$$

$$= nx^n f\left(\frac{y}{x}\right)$$

$= nf$ [using (1)]

$$\therefore x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf.$$

(1)

Note: If z be a homogeneous function of x, y of degree n then $x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=n(n-1)z$.

Proof: We know that by Euler's Theorem, If z be a homogeneous function of x, y of degree n, then

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz \quad \text{--- (1)}$$

Differentiating (2) partially w.r.t 'x' we have

$$x\frac{\partial^2 z}{\partial x^2}+\frac{\partial z}{\partial x}+y\frac{\partial^2 z}{\partial x\partial y}=n\frac{\partial z}{\partial x} \quad \text{--- (ii)}$$

Again, Differentiating (2) partially w.r.t 'y' we have

$$x\frac{\partial^2 z}{\partial y\partial x}+y\frac{\partial^2 z}{\partial y^2}+\frac{\partial z}{\partial y}=n\frac{\partial z}{\partial y} \quad \text{--- (iii)}$$

Now, multiplying (ii), (iii) by x, y respectively and adding, we get

$$x^2\frac{\partial^2 z}{\partial x^2}+x\frac{\partial z}{\partial x}+xy\frac{\partial^2 z}{\partial x\partial y}+xy\frac{\partial^2 z}{\partial y\partial x}+y^2\frac{\partial^2 z}{\partial y^2}+y\frac{\partial z}{\partial y}=nx\frac{\partial z}{\partial x}+ny\frac{\partial z}{\partial y}$$

$$\Rightarrow x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=n(n-1)z.$$

Ex: (1) If $u=x^2\tan^{-1}\frac{y}{x}-y^2\tan^{-1}\frac{x}{y}$ then prove that $\frac{\partial^2 u}{\partial x\partial y}=\frac{x^2-y^2}{x^2+y^2}$

Solution: Given $u=x^2\tan^{-1}\frac{y}{x}-y^2\tan^{-1}\frac{x}{y}$

$$\therefore \frac{\partial u}{\partial y}=x^2\frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x}+y^2\frac{1}{1+\frac{x^2}{y^2}}\cdot\frac{x}{y^2}-2y\tan^{-1}\frac{x}{y}$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{x^3}{x^2+y^2} + \frac{xy^2}{x^2+y^2} - 2y\tan^{-1}\frac{x}{y}$$

$$\Rightarrow \frac{\partial u}{\partial y} = x - 2y\tan^{-1}\frac{x}{y}.$$

Again, $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$

$$= \frac{\partial}{\partial x}\left(x - 2y\tan^{-1}\frac{x}{y}\right)$$

$$= 1 - 2y\frac{1}{1+\frac{x^2}{y^2}}\frac{1}{y}$$

$$= 1 - \frac{2y^2}{y^2+x^2} = \frac{x^2-y^2}{x^2+y^2}.$$

Ex ②: If $u = \frac{1}{\sqrt{x^2+y^2+z^2}}$ show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad \text{(VVI)}$$

Solution: Given $u = (x^2+y^2+z^2)^{-\frac{1}{2}}$

$$\frac{\partial u}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}(2x) = -x(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left[-x(x^2+y^2+z^2)^{-\frac{3}{2}}\right]$$

$$= -x\left[-\frac{3}{2}(x^2+y^2+z^2)^{-\frac{5}{2}}(2x)\right] + (x^2+y^2+z^2)^{-\frac{3}{2}}(-1)$$

$$= 3x^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$$

Similarly

$$\frac{\partial^2 u}{\partial y^2} = 3y^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial^2 u}{\partial z^2} = 3z^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$$

Now, $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$

$$= 3(x^2+y^2+z^2)^{-5/2}(x^2+y^2+z^2) - 3(x^2+y^2+z^2)^{-3/2}$$

$$= 3(x^2+y^2+z^2)^{-3/2} - 3(x^2+y^2+z^2)^{-3/2}$$

$$= 0$$

//

Ex(3): If $u = \tan^{-1}\frac{x^3+y^3}{x-y}$ prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \sin 2u$$

Solution: To apply Euler's theorem, first we have to check; whether u is homogeneous function or not?

$$u(\lambda x, \lambda y) = \tan^{-1}\frac{(\lambda x)^3+(\lambda y)^3}{\lambda x - \lambda y}$$

$$= \tan^{-1}\frac{\lambda^2(x^3+y^3)}{x-y} \neq \lambda^2 u(x,y)$$

Clearly, ~~$u(\lambda x, \lambda$~~ u is not homogeneous.

But, The values of u can be re-written as

$$u = \tan^{-1} z \quad , \quad \text{let } z = \frac{x^3+y^3}{x-y}$$

$$\Rightarrow z = \tan u$$

Clearly, z is homogeneous degree of 2.

$\therefore$ By Euler's theorem, we have

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y} = 2z$$

$$\Rightarrow x\frac{\partial}{\partial x}(\tan u)+y\frac{\partial}{\partial y}(\tan u) = 2\tan u$$

$$\Rightarrow x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2\tan u$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 2\sin u\cos u \qquad \left[\because \sec\theta = \frac{1}{\cos\theta}\right]$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \sin 2u \qquad [\because \sin 2\theta = 2\sin\theta\cos\theta]$$

Ex(4) Verify Euler's Theorem for

(i) $z = ax^2+2hxy+by^2$ (ii) $z = (x^2+xy+y^2)$

(iii) $z = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$ (iv) $z = x^n \log\frac{y}{x}$

(v) $z = \dfrac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}$

Solution: (i) Given $z(x,y) = ax^2+2hxy+by^2$

$$\therefore z(\lambda x,\lambda y) = a(\lambda x)^2+2h(\lambda x)(\lambda y)+b(\lambda y)^2$$
$$= \lambda^2(ax^2+2hxy+by^2)$$
$$= \lambda^2 z(x,y)$$

$\therefore$ z is a homogeneous function of degree 2.

By Euler's Theorem, we have

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y} = 2z.$$

(iii) $z(x,y) = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$

$$\therefore z(\lambda x,\lambda y) = \sin^{-1}\frac{\lambda x}{\lambda y} + \tan^{-1}\frac{\lambda y}{\lambda x}$$
$$= \lambda^0 (z(\lambda x,\lambda y))$$

$\therefore$ z is a homogeneous function of degree 0.

$$\therefore \quad x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y} = 0.$$

(iv) Given $z(x,y) = x^n \log \frac{y}{x}$

$\therefore z(\lambda x, \lambda y) = (\lambda x)^n \log \frac{\lambda y}{\lambda x}$

$= \lambda^n x^n \log \frac{y}{x}$

$= \lambda^n z(x,y)$

$\therefore$ z is a homogeneous function of degree n.

Therefore, by Euler's Theorem, we have

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz.$$

Ex(5): If $u = f(y/x)$ show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

Solution: In this question, we can proved by two methods.

Method (1): Given $u = f\left(\frac{y}{x}\right)$

$\therefore \frac{\partial u}{\partial x} = f'\left(\frac{y}{x}\right)\frac{\partial}{\partial x}\left(\frac{y}{x}\right) = -\frac{y}{x^2} f'\left(\frac{y}{x}\right)$

and $\frac{\partial u}{\partial y} = f'\left(\frac{y}{x}\right)\frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{1}{x} f'\left(\frac{y}{x}\right)$

Now, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{y}{x} f'\left(\frac{y}{x}\right) + \frac{y}{x} f'\left(\frac{y}{x}\right)$

$= 0$

\#

Method (2): Given $u = f\left(\frac{y}{x}\right)$

$\therefore u(\lambda x, \lambda y) = f\left(\frac{\lambda y}{\lambda x}\right) = f\left(\frac{y}{x}\right)$

$= \lambda^0 f\left(\frac{y}{x}\right)$

$= \lambda^0 u$

$\therefore$ u is a homogeneous function of degree 0.
By Euler's theorem, we get

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0.$$

Ex(6): If $z=xy\,f\left(\frac{x}{y}\right)$, show that

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2z.$$

Ans: [Try yourself] [Hint: z is a homogeneous function of degree 2]

Ex(7): If $u=\log\frac{x^2+y^2}{x+y}$, prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=1$$

Ex(8): If $u=\sin^{-1}\frac{x^2+y^2}{x+y}$, show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u.$$

Solution: [Ex(7) and Ex(8); Solving Technique are same]

Given, $u=\sin^{-1}\frac{x^2+y^2}{x+y}$

Clearly u is not homogeneous function, (?)

$\therefore$ we can re-write u as below

$$u=\sin^{-1}z,\quad \text{let } z=\frac{x^2+y^2}{x+y}$$

$$\Rightarrow z=\sin u$$

Clearly, z is homogeneous function of degree 1. Therefore by Euler's theorem

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z$$

$\Rightarrow x\frac{\partial}{\partial x}(\sin u)+y\frac{\partial}{\partial y}(\sin u)=\sin u$

$\Rightarrow x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}=\sin u$

$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u$.

Ex(9): If $u=\sin^{-1}\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$ show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$$

Ex(10): If $z=f(x+ay)+\phi(x-ay)$

prove that $\frac{\partial^2 z}{\partial y^2}=a^2\frac{\partial^2 z}{\partial x^2}$

Ex(11): If $u=f(r)$ where $r=\sqrt{x^2+y^2}$

prove that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f''(r)+\frac{1}{r}f'(r)$

Ex(12): If $u=\log(x^2+y^2+z^2)$, prove that

$$x\frac{\partial^2 u}{\partial y\partial z}=\frac{\partial^2 u}{\partial z\partial x}=z\frac{\partial^2 u}{\partial x\partial y}$$

Ex(13): If $V=r^m$, where $r^2=x^2+y^2+z^2$

Show that $\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=m(m+1)r^{m-2}$

Ex(14): If $u=\tan^{-1}\frac{x^3+y^3}{x+y}$, find

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}$$

[Ans: $(1-4\sin^2 u)\sin 2u$]

Ex (15): If $u = \sin^{-1}\left\{\dfrac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}\right\}^{1/2}$ show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{144}\left(13+\tan^2 u\right)$$

Solution: Given $u = \sin^{-1}\left\{\dfrac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}\right\}^{1/2}$

$$\Rightarrow u = \sin^{-1} z \quad \text{(Say)}$$

$$\Rightarrow z = \sin u \quad \text{——— (1)}$$

Now, $\dfrac{\partial z}{\partial x} = \cos u \dfrac{\partial u}{\partial x}$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} = \cos u \frac{\partial^2 u}{\partial x^2} - \sin u\left(\frac{\partial u}{\partial x}\right)^2$$

$$\therefore \frac{\partial^2 z}{\partial y\partial x} = \cos u \frac{\partial^2 u}{\partial y\partial x} - \sin u \frac{\partial u}{\partial y}\cdot\frac{\partial u}{\partial x}.$$

And, $\dfrac{\partial z}{\partial y} = \cos u \dfrac{\partial u}{\partial y}$

$$\Rightarrow \frac{\partial^2 z}{\partial y^2} = \cos u \frac{\partial^2 u}{\partial y^2} - \sin u\left(\frac{\partial u}{\partial y}\right)^2$$

As z is homogeneous function of x, y of degree $-\frac{1}{12}$, ∴ By Note (1), we have [Above Ex (1)]

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = -\frac{1}{12}\left(-\frac{1}{12}-1\right)z$$

$$\Rightarrow x^2\left[\cos u \frac{\partial^2 u}{\partial x^2} - \sin u\left(\frac{\partial u}{\partial x}\right)^2\right] + 2xy\left[\cos u\frac{\partial^2 u}{\partial y\partial x} - \sin u\frac{\partial u}{\partial y}\cdot\frac{\partial u}{\partial x}\right] + y^2\left[\cos u\frac{\partial^2 u}{\partial y^2} - \sin u\left(\frac{\partial u}{\partial y}\right)^2\right] = \frac{13}{144}\sin u$$

$$\Rightarrow \cos u\left[x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}\right] - \sin u\left[x^2\left(\frac{\partial u}{\partial x}\right)^2 + 2xy\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+y^2\left(\frac{\partial u}{\partial y}\right)^2\right] = \frac{13}{144}\sin u$$

$$\Rightarrow \cos u\left[x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}\right] - \sin u\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]^2 = \frac{13}{144}\sin u \quad \text{——} (*)$$

<u>Now,</u> $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = -\frac{1}{12}\tan u$ (why ?)

$$\therefore (*) \Rightarrow \cos u\left[x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}\right] = \frac{\sin u}{144}\tan^2 u + \frac{13}{144}\sin u$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{144}\left(13+\tan^2 u\right).$$

$\#$